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Nilpotent Lie algebras with 2-dimensional commutator ideals[☆]

C. Bartolone *, A. Di Bartolo, G. Falcone

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, I-90123 Palermo, Italy

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We classify all (finitely dimensional) nilpotent Lie k-algebras $\mathfrak{h}$ with 2-dimensional commutator ideals $\mathfrak{h}'$, extending a known result to the case where $\mathfrak{h}'$ is non-central and k is an arbitrary field. It turns out that, while the structure of $\mathfrak{h}$ depends on the field k if $\mathfrak{h}'$ is central, it is independent of k if $\mathfrak{h}'$ is non-central and is uniquely determined by the dimension of $\mathfrak{h}$. In the case where k is algebraically or real closed, we also list all nilpotent Lie k-algebras $\mathfrak{h}$ with 2-dimensional central commutator ideals $\mathfrak{h}'$ and $\dim_{\mathsf{k}} \mathfrak{h} \leqslant 11$.

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1. Introduction

For the classical cases $\mathsf{k} = \mathbb{C}$ and $\mathsf{k} = \mathbb{R}$ complete lists of non-isomorphic nilpotent k-algebras are known in the lower dimensions. Morozov in [5] and Vranceanu in [7], independently, classify all nilpotent complex Lie algebras of dimension $\leqslant 6$, whereas Ancochea Bermudez and Goze extend in [1] such a classification to include the dimension 7. For the real case Diximer gives in [3] a correct and complete list of non-isomorphic real nilpotent Lie algebras of dimension $\leqslant 6$.

A different point of view is to consider nilpotent Lie algebras $\mathfrak{h}$ with commutator ideals $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ of small dimension. The structure of a nilpotent Lie algebra $\mathfrak{h}$ is straightforward when the dimension of $\mathfrak{h}'$ is one, $\mathfrak{h}$ being the direct sum of a Heisenberg algebra and an Abelian Lie algebra. When $\dim_{\mathsf{k}} \mathfrak{h}' = 2$,

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* Corresponding author.
E-mail addresses: cg@math.unipa.it (C. Bartolone), alfonso@math.unipa.it (A. Di Bartolo), gfalcone@unipa.it (G. Falcone).

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the classification of pairs of alternating forms (see [6]) supplies an effective tool, even in characteristic 2. Apparently, Gauger [4] was the first to use the canonical reduction of a pair of alternating forms in classifying nilpotent Lie algebras over an algebraically closed field of characteristic $\neq 2$ with 2-dimensional central commutator ideals.

The topic was the object of a recent paper of Belitskii et al. [2], where the authors prove that a similar approach is in general not possible when commutator ideals have dimension three. Moreover, they give another elegant description of complex nilpotent Lie algebras with 2-dimensional central commutator ideals.

In this paper, we consider the following two variations. First, we extend Gauger's classification to the case of an arbitrary field k (with sufficiently many elements). In particular, in the case where the field k is real closed, we give a table portraying all such algebras $\mathfrak{h}$ for $\dim_{\mathsf{k}} \mathfrak{h} \leqslant 11$, from which one infers the analogous list for the algebraic closed case.

Second, we stress the fact that, by simply imposing Jacobi's identity, we can also classify nilpotent Lie k-algebras with 2-dimensional non-central commutator ideals. In particular, it turns out (Theorem 5) that, if we are given a finitely dimensional nilpotent Lie algebra $\mathfrak{h}$ over an arbitrary field k having 1-dimensional centre and 2-dimensional commutator ideal, then the structure of $\mathfrak{h}$ is completely determined by its dimension and is independent of the field k.

2. Nilpotent Lie algebras of $(l, 2, n)$-type

This section extends Section 4 in [2] to the case of an arbitrary field with sufficiently many elements, using essentially the same argument.

For a given nilpotent Lie algebra $\mathfrak{h}$ of finite dimension over a field k, with centre $\mathfrak{z}(\mathfrak{h})$ and commutator ideal $\mathfrak{h}'$, we set

$$l := \dim_{\mathsf{k}} \mathfrak{h}' \cap \mathfrak{z}(\mathfrak{h}), \quad m := \dim_{\mathsf{k}} \mathfrak{h}', \quad n := \dim_{\mathsf{k}} \mathfrak{h} / \left(\mathfrak{h}' + \mathfrak{z}(\mathfrak{h})\right).$$

Nilpotent Lie k-algebras of *type* (l, m, n) with $m = 2$ will be thoroughly considered in this paper.

Let U be an n-dimensional linear subspace of $\mathfrak{h}$ complementary to the ideal $\mathfrak{h}' + \mathfrak{z}(\mathfrak{h})$:

$$\mathfrak{h} = \left(\mathfrak{h}' + \mathfrak{z}(\mathfrak{h})\right) \oplus U.$$

Then, for any basis $\{x, y\}$ of $\mathfrak{h}'$, there are two alternating forms $\Xi, \Upsilon : U \times U \to \mathsf{k}$ defined by putting

$$[u_1, u_2] = \Xi(u_1, u_2)x + \Upsilon(u_1, u_2)y, \tag{1}$$

for all $u_1, u_2 \in U$. Canonical matrices (X, Y) representing the pair (Ξ, Υ) can be obtained as follow (see for instance [6]): (X, Y) is a direct sum, uniquely determined up to permutation of the summands, of pairs of the form

$$(L_t, M_t)^{\triangledown}, \quad (I_t, K(f^{\alpha}))^{\triangledown}, \quad (J_t(0), I_t)^{\triangledown},$$

where

$$(A, B)^{\triangledown} := \left(\begin{pmatrix} 0 & -A \\ {}^{\mathrm{t}}A & 0 \end{pmatrix}, \begin{pmatrix} 0 & -B \\ {}^{\mathrm{t}}B & 0 \end{pmatrix} \right),$$

$$L_t := \begin{pmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \; M_t := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix} \quad (t - \text{by} - (t+1)),$$

I_t is the $t \times t$ identity matrix, $J_t(0)$ is the $t \times t$ singular Jordan block and $K(f^{\alpha})$ is the companion matrix of a power of a monic irreducible polynomial f over k with $\alpha \deg f = t$. Such a canonical pair (X, Y) depends, of course, on the chosen basis of $\mathfrak{h}'$: we shall always refer to bases $\{x, y\}$ of $\mathfrak{h}'$ with $x \in \mathfrak{h}' \cap \mathfrak{z}(\mathfrak{h})$. If $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are such two bases and

$$x_1 = C_{11}x_2 + C_{12}y_2, \; y_1 = C_{21}x_2 + C_{22}y_2$$

with $C := (C_{ij}) \in \mathrm{GL}_2(\mathsf{k})$ and $C_{12} = 0$ if $\dim_{\mathsf{k}} \left(\mathfrak{h}' \cap \mathfrak{z}(\mathfrak{h})\right) = 1$, then from (1) we infer the relationship between the corresponding canonical pairs (X_1, Y_1) and (X_2, Y_2):

$$X_1C_{11} + Y_1C_{21} = PX_2{}^{\mathrm{t}}P, \quad X_1C_{12} + Y_1C_{22} = PY_2{}^{\mathrm{t}}P, \tag{2}$$

for some $P \in \mathrm{GL}_n(\mathsf{k})$. In particular, transformations (2) with $C_{11} = C_{22} = 1$ and $C_{12} = 0$ allow one to take canonical forms (X, Y) with X having no summands of the form $(J_t(0), I_t)$, provided the cardinality of the field k is greater than the number of summands of the form $(I_t, K(f^\alpha))^\triangledown$, or $(J_t(0), I_t)^\triangledown$. Thus we have

Theorem 1. *Any nilpotent Lie k-algebra $\mathfrak{h}$ of $(l, 2, n)$-type with $n < 2|\mathsf{k}|$ decomposes, as a vector space, into*

$$\mathfrak{h} = \left(\mathfrak{h}' + \mathfrak{z}(\mathfrak{h})\right) \oplus \mathfrak{v} \oplus \mathfrak{w}$$

for some pair $\mathfrak{v}$ and $\mathfrak{w}$ of Abelian subalgebras of $\mathfrak{h}$ with $r := \dim_{\mathsf{k}} \mathfrak{v} \leqslant s := \dim_{\mathsf{k}} \mathfrak{w}$. More precisely, there exist bases $\{x, y\}$ of $\mathfrak{h}'$ with $x \in \mathfrak{h}' \cap \mathfrak{z}(\mathfrak{h})$, $\{v_1, \ldots, v_r\}$ of $\mathfrak{v}$ and $\{w_1, \ldots, w_s\}$ of $\mathfrak{w}$ such that

$$[v_i, w_j] = X_{ij}x + Y_{ij}y \qquad (i = 1, \ldots, r; j = 1, \ldots, s)$$

with $X := (X_{ij})$ and $Y := (Y_{ij})$ a pair of matrices such that

$$(X, Y) = \bigoplus_{k=1}^{e} (L_{t_k}, M_{t_k}) \bigoplus_{k=e+1}^{d} \left(I_{t_k}, K\left(f_k^{\alpha_k}\right)\right). \tag{3}$$

Let k be algebraically closed, then $f_k(\mathrm{T}) = \mathrm{T} - a_k$ for some $a_k \in \mathsf{k}$ and $K\left(f_k^{\alpha_k}\right)$ can be replaced by the Jordan block $J_{t_k}(a_k)$, i.e.

$$(X, Y) = \bigoplus_{k=1}^{e} (L_{t_k}, M_{t_k}) \bigoplus_{k=e+1}^{d} \left(I_{t_k}, J_{t_k}(a_k)\right). \tag{4}$$

Let k be real closed, then either $f_k(\mathrm{T}) = \mathrm{T} - a_k$ for some $a_k \in \mathsf{k}$ and $K\left(f_k^{\alpha_k}\right)$ can be replaced by the real Jordan block $J_{t_k}(a_k)$, or $f_k(\mathrm{T}) = \mathrm{T}^2 - 2a_k'\mathrm{T} + {a_k'}^2 + {a_k''}^2$ for some $a_k := a_k' + \sqrt{-1}\, a_k''$ in the algebraic closure $\bar{\mathsf{k}}$ of k, with $a_k'' \neq 0$, and $K\left(f_k^{\alpha_k}\right)$ can be replaced by the "realification"

$$\widetilde{J_{t_k}(a_k)} := \begin{pmatrix} a_k' & -a_k'' & 1 & 0 & & & & \\ a_k'' & a_k' & 0 & 1 & & & & \\ & & \ddots & & \ddots & & & \\ & & & \ddots & & \ddots & & \\ & & & & a_k' & -a_k'' & 1 & 0 \\ & & & & a_k'' & a_k' & 0 & 1 \\ & & & & & & a_k' & -a_k'' \\ & & & & & & a_k'' & a_k' \end{pmatrix} \in \mathrm{GL}_{t_k}(\mathsf{k})$$

of the Jordan matrix

$$\left(\begin{array}{c|c} J_{\frac{t_k}{2}}\left(a_k' + \sqrt{-1}a_k''\right) & \\ \hline & J_{\frac{t_k}{2}}\left(a_k' - \sqrt{-1}a_k''\right) \end{array}\right) \in \mathrm{GL}_{t_k}\left(\bar{\mathsf{k}}\right),$$

i.e.

$$(X, Y) = \bigoplus_{k=1}^{e} (L_{t_k}, M_{t_k}) \bigoplus_{k=e+1}^{e+q} \left(I_{t_k}, J_{t_k}(a_k)\right) \bigoplus_{k=e+q+1}^{d} \left(I_{t_k}, \widetilde{J_{t_k}(a_k)}\right). \tag{5}$$

Remark 2. As we pointed out above, the canonical pair (X, Y) given in Theorem 1 is uniquely determined up to a permutation of the indices and a change of basis of $\mathfrak{h}'$. While we can settle the summands (L_{t_k}, M_{t_k}) by putting the integers t_k in (weakly) decreasing order, this is not enough for the remaining summands. For instance, if k is algebraically closed, the isomorphism condition (2) says that we can replace each Jordan block $J_t(a)$ with eigenvalue a by the Jordan block $J_t(b)$ with the eigenvalue b defined by the k-linear fractional transformation

$$b := \frac{C_{21} + aC_{22}}{C_{11} + aC_{12}}. \tag{6}$$

Likewise, if k is real closed, we can replace both Jordan blocks with eigenvalue a in k and realifications of Jordan blocks with eigenvalue a in the algebraic closure $\bar{\mathsf{k}}$ using k-linear fractional transformations such as in (6).

3. Nilpotent Lie algebras of $(2, 2, n)$-type

Obviously in the case where $\mathfrak{h}' \subseteq \mathfrak{z}(\mathfrak{h})$ Theorem 1 completely determines the structure of $\mathfrak{h}$. In order to distinguish the isomorphism classes of such nilpotent Lie algebras, we are going to introduce some notation covering the cases where k is algebraically or real closed.

If k is algebraically closed we shall use the sequence

$$p^{t_1} \dots p^{t_e} a_{e+1}^{t_{e+1}} \dots a_d^{t_d}$$

to represent the isomorphism class of the nilpotent Lie k-algebra of $(2, 2, n)$-type the structure of which is expressed by the pair of alternating matrices in (4). This can be well-defined by arranging the indices $1, \dots, e$ so that $t_1 \geqslant \dots \geqslant t_e$, whereas the indices $e+1, \dots, d$ can be arranged as follows: denote by $\mu(a)$ the number of Jordan blocks with eigenvalue $a \in \mathsf{k}$ occurring in Y, then we set $e+1 \leqslant i \leqslant j \leqslant d$ precisely if one of the following holds:

(i) $a_i = a_{i+1} = \dots = a_{j-1} = a_j$ and $t_i \geqslant t_{i+1} \geqslant \dots \geqslant t_{j-1} \geqslant t_j$;
(ii) $a_i \neq a_j$ and $\mu(a_i) \geqslant \mu(a_j)$;
(iii) $a_i \neq a_j$, $\mu(a_i) = \mu(a_j)$ and $t_i \geqslant t_j$.

In the case where k is real closed the canonical decomposition (5) suggests to represent the corresponding isomorphism class by the sequence

$$p^{t_1} \dots p^{t_e} a_{e+1}^{t_{e+1}} \dots a_{e+q}^{t_{e+q}} \left(a'_{e+q+1} + \sqrt{-1}a''_{e+q+1}\right)^{t_{e+q+1}} \dots \left(a'_d + \sqrt{-1}a''_d\right)^{t_d},$$

where the indices are arranged similarly to the case where k is algebraically closed. As we pointed out in Remark 2, while the ordered sequence $p^{t_1}, \dots, p^{t_d}$ is characteristic for the isomorphism class of a nilpotent Lie algebra of $(2, 2, n)$-type, the sequence of eigenvalues, ordered as above, can be changed, according to the isomorphism condition (2), by means of k-linear fractional transformations; more precisely we have

Theorem 3. *Let k be an algebraically or real closed field. Two ordered sequences $p^{t_1} \dots p^{t_e} a_{e+1}^{t_{e+1}} \dots a_d^{t_d}$ and $p^{t_1} \dots p^{t_e} b_{e+1}^{t_{e+1}} \dots b_d^{t_d}$, where for $e+q < k \leqslant d$ the elements a_k, b_k are taken in the algebraic closure $\bar{\mathsf{k}}$ if $\mathsf{k} \neq \bar{\mathsf{k}}$, represent the same isomorphism class of nilpotent Lie k-algebras of $(2, 2, n)$-type if and only if there exists a k-linear fractional transformation $\sigma : \mathsf{k} \to \mathsf{k}$ such that $b_k = \sigma(a_k)$ for all $k = e+1, \dots, d$.*

Remark 4. As the projective linear group is 3-transitive on the projective line, Theorem 3 says that, if there are eigenvalues $a_i \in \mathsf{k}$ we can always assume $a_1 = 0, a_2 = 1$ (if at least two distinct eigenvalues occur) and $a_3 = -1$ (if at least three distinct eigenvalues occur). Notice that the sequence $a^1 \dots a^1$ represents no nilpotent algebra of $(2, 2, n)$-type: in fact, we could replace the eigenvalue a by 0 and the commutator ideal would be 1-dimensional. Furthermore, if k is real closed and $d \geqslant e+q+1$, we may take $a_{e+q+1} = \sqrt{-1}$, and, if we have in addition $q \neq 0$, we may take $a_{e+1} = 0$. This allows one, in low dimension, to list quickly the isomorphism classes of nilpotent Lie k-algebras $\mathfrak{h}$ of $(2, 2, n)$-type with $\mathfrak{z}(\mathfrak{h}) = \mathfrak{h}'$ for the cases where k is algebraically or real closed. Below you find the complete list for k real closed and $\dim_{\mathsf{k}} \mathfrak{h} \leqslant 11$.

$n = \dim_{\mathsf{k}} \mathfrak{h}/\mathfrak{z}(\mathfrak{h})$	Isomorphism classes
3	p
4	$0^2 \quad 01 \quad \sqrt{-1}$
5	$p^2 \quad p\,0$
6	$pp \quad 0^3 \quad 0^2 0 \quad 0^2 1 \quad 001 \quad 01(-1) \quad 0\sqrt{-1}$
7	$p^3 \quad p^2 0 \quad p\,0^2 \quad p\,00 \quad p\,01 \quad p\sqrt{-1}$
8	$p^2 p \quad pp\,0 \quad 0^4 \quad 0^3 0 \quad 0^2 0^2 \quad 0^3 1 \quad 0^2 01 \quad 0001 \quad 0^2 1(-1)$
	$001(-1) \quad 01(-1)a \quad 0^2\sqrt{-1} \quad 00\sqrt{-1} \quad 01\sqrt{-1} \quad \sqrt{-1}^2$
	$\sqrt{-1}\sqrt{-1} \quad \sqrt{-1}\left(a+\sqrt{-1}\right) \quad \sqrt{-1}\left(a-\sqrt{-1}\right)$
9	$p^4 \quad ppp \quad p^3 0 \quad p^2 0^2 \quad p^2 00 \quad p^2 01 \quad p^2\sqrt{-1} \quad p\,0^3 \quad p\,0^2 0$
	$p\,000 \quad p\,0^2 1 \quad p\,001 \quad p\,01(-1) \quad p\,0\sqrt{-1}$

The list above contains finitely many isomorphism classes, apart from the parameterized classes $01(-1)a$, with a in k, and $\sqrt{-1}\,(a \pm \sqrt{-1})$ with a a square in k. For $n \geqslant 10$ infinitely many isomorphism classes always occur. Notice that a Lie algebra of the class 01 splits into the direct sum of two 3-dimensional Lie algebras and corresponds to $2L_2$ in Morozov's list A [5], whereas 6-dimensional Lie algebras belonging to 0^2 and $\sqrt{-1}$ are, respectively, the algebras 4 and 5 in Morozov's list B.

Manifestly we obtain the analogous list for the algebraic closure $\bar{\mathsf{k}}$ of k by deleting the isomorphic classes in the above list where some $\sqrt{-1}$ occurs. Looking at the classification of 7-dimensional complex nilpotent Lie algebras given in [1], we see that p^2 and $p\,0$ are, respectively, the isomorphism classes of the Lie algebra n^7_{127} and n^7_{128} listed there.

4. Nilpotent Lie algebras of $(1, 2, n)$-type

Assume now $\mathfrak{h}$ is a nilpotent Lie k-algebra of $(1, 2, n)$-type with $\dim_{\mathsf{k}} \mathfrak{z}(\mathfrak{h}) = 1$. We shall prove that the structure of $\mathfrak{h}$ is already determined by its dimension.

Consider the basis $\{x, y, v_1, \ldots, v_r, w_1, \ldots, w_s\}$ of $\mathfrak{h}$ giving the canonical pairs of alternating matrices in (3). Then Theorem 1 claims that we know the structure of $\mathfrak{h}$ whenever we determine all the brackets $[y, v_i]$ and $[y, w_j]$, which are vectors of $\langle x \rangle_{\mathsf{k}} = \mathfrak{z}(\mathfrak{h})$. Thus, in order to impose Jacobi's identity, we have to point out the brackets giving vectors having a nontrivial component in the subspace $\langle y \rangle_{\mathsf{k}}$. Use a double index for the vectors $v_1, \ldots, v_r$ (resp. $w_1, \ldots, w_s$) according to the decomposition (3), i.e.

$$\begin{array}{llll} v_{kl}, & 1 \leqslant k \leqslant d, & 1 \leqslant l \leqslant t_k; & \\ w_{kl}, & 1 \leqslant k \leqslant d, & 1 \leqslant l \leqslant t_k \text{ for } k > e, & 1 \leqslant l \leqslant t_k + 1 \text{ for } k \leqslant e. \end{array}$$

Then, by Theorem 1, the only brackets that can give vectors having a nontrivial component in $\langle y \rangle_{\mathsf{k}}$ are:

$$\begin{array}{lllll} \left[v_{kl}, w_{k,\,l+1}\right], & \left[w_{k,\,l+1}, v_{k\,l}\right], & 1 \leqslant k \leqslant e, & 1 \leqslant l \leqslant t_k; \\ \left[v_{kl}, w_{k,\,l+1}\right], & \left[w_{k,\,l+1}, v_{k\,l}\right], & e+1 \leqslant k \leqslant d & 1 \leqslant l \leqslant t_k - 1; \\ \left[v_{kt_k}, w_{k\,l}\right], & \left[w_{kl}, v_{k\,t_k}\right], & e+1 \leqslant k \leqslant d, & 1 \leqslant l \leqslant t_k, \ f_k(\mathrm{T}) \neq \mathrm{T}. \end{array} \tag{7}$$

Consequently, the unique (non-ordered) triples of elements of $\mathfrak{h}$ for which Jacobi's identity could give conditions are:

$$\begin{array}{llll} \left(v_{kl}, w_{k,\,l+1}, v_{k'l'}\right), & \left(w_{k,\,l+1}, v_{k\,l}, w_{k'l'}\right), & 1 \leqslant k \leqslant e, & 1 \leqslant l \leqslant t_k; \\ \left(v_{kl}, w_{k,\,l+1}, v_{k'l'}\right), & \left(w_{k,\,l+1}, v_{kl}, w_{k'l'}\right), & e+1 \leqslant k \leqslant d, & 1 \leqslant l \leqslant t_k - 1; \\ \left(v_{kt_k}, w_{k\,l}, v_{k'l'}\right), & \left(w_{kl}, v_{k\,t_k}, w_{k'l'}\right), & e+1 \leqslant k \leqslant d, & 1 \leqslant l \leqslant t_k, \ f_k(\mathrm{T}) \neq \mathrm{T}. \end{array} \tag{8}$$

Let

$$[y, v_{k'l'}] = \gamma_{k'l'}x, \quad [y, w_{k'l'}] = \delta_{k'l'}x$$

and let (k, l) a pair such as in (7). Then, looking at (8) we infer $\gamma_{k'l'} = \delta_{k'l'} = 0$ for any pair (k', l') with $k' \neq k$, which means that no two of the pairs (k, l) given in (7) can occur for different values of k. This reduces matters to the following two cases:

$$\begin{array}{llll} \text{(i)} & e = 1, & t_k = 1\ \forall k > 1, & f_k(\mathrm{T}) = \mathrm{T}\ \forall k > 1; \\ \text{(ii)} & e = 0, & t_k = 1\ \forall k > 1, & f_k(\mathrm{T}) = \mathrm{T}\ \forall k > 1. \end{array} \tag{9}$$

Assume that (9.i) holds. If $t_1 > 1$, then for any pair of distinct integers $l, l' \in \{1, \ldots, t_1\}$ we consider the triples $(v_{1,l}, w_{1,l+1}, v_{1,l'})$ and $(v_{1,l'}, w_{1,l'+1}, v_{1,l})$ (resp. $(w_{1,l+1}, v_{1,l}, w_{1,l'})$ and $(w_{1,l'+1}, v_{1,l'}, w_{1,l})$): Jacobi's identity for such triples gives both $\gamma_{1\,l'} = 0$ and $\gamma_{1\,l} = 0$ (resp. $\delta_{1\,l'} = 0$ and $\delta_{1\,l} = 0$). Besides, the triple $(w_{1,2}, v_{1,1}, w_{1,t_1+1})$ gives $\delta_{1,t_1+1} = 0$.

Thus, in the case (9.i) we also have $t_1 = 1$. Furthermore, by applying Jacobi's identity to the triple (w_{12}, v_{11}, w_{11}) we get $\delta_{11} = 0$.

Let now $e = 0$ and assume $t_1 > 1$. If $t_1 > 2$, for any pair of distinct integers $l, l' \in \{1, \ldots, t_1 - 1\}$ we obtain as in the previous case both $\gamma_{1\,l'} = 0$ and $\gamma_{1\,l} = 0$ (resp. $\delta_{1\,l'} = 0$ and $\delta_{1\,l} = 0$). Furthermore, from the triples $(v_{11}, w_{12}, v_{1\,t_1})$ and $(w_{12}, v_{11}, w_{1\,t_1})$ follow $\gamma_{1\,t_1} = \delta_{1\,t_1} = 0$, respectively. Thus, we have $t_1 \leqslant 2$ in the case (9.ii).

Let $e = 0$ with $t_1 = 2$, then $[v_{11}, w_{11}] = x$. If $f_1(\mathrm{T}) \neq \mathrm{T}$, $[v_{12}, w_{11}]$ and $[v_{11}, w_{12}]$ are nonzero vectors in $\langle y \rangle_{\mathsf{k}}$ and, via Jacobi's identity, we obtain $\gamma_{11} = 0$ from the triple (v_{12}, w_{11}, v_{11}), $\delta_{11} = 0$ from (w_{12}, v_{11}, w_{11}) and, consequently, $\gamma_{12} = 0$ from (v_{11}, w_{12}, v_{12}) and $\delta_{12} = 0$ from (w_{11}, v_{12}, w_{12}). Therefore, $f_1(\mathrm{T}) = \mathrm{T}$ in case $t_1 = 2$. Besides, in such a case, we infer $\gamma_{12} = 0$ and $\delta_{11} = 0$ by applying Jacobi's identity to the triples (v_{11}, w_{12}, v_{12}) and (w_{12}, v_{11}, w_{11}), respectively.

Summing up, we have $\gamma_{kl} = \delta_{kl} = 0\ \forall\ k, l$, apart from the following cases:

$$\begin{array}{llllll} 1. & \gamma_{11},\ \delta_{12}, & e = 1, & t_k = 1\ \forall k, & & f_k(\mathrm{T}) = \mathrm{T}\ \forall k > 1; \\ 2. & \gamma_{11},\ \delta_{12}, & e = 0, & t_1 = 2,\ t_k = 1\ \forall k > 1, & & f_k(\mathrm{T}) = \mathrm{T}\ \forall k; \\ 3. & \gamma_{d1},\ \delta_{d1}, & e = 0, & t_k = 1\ \forall k, & & f_k(\mathrm{T}) = \mathrm{T}\ \forall k < d,\ f_d(\mathrm{T}) \neq \mathrm{T}. \end{array} \tag{10}$$

The Lie algebra defined through (10.2) can be excluded in view of the fact that $y + \gamma_{11}w_{11} - \delta_{12}v_{12}$ would be an element of $\mathfrak{z}(\mathfrak{h})$, linearly independent of x.

Assume (10.1) holds. If $\delta_{12} = 0$, then $y + \gamma_{11}w_{11}$ is a further central element. So, we may assume $\delta_{12} \neq 0$ and the replacements

$$v_{11} \mapsto \delta_{12}v_{11} - \gamma_{11}w_{12},\ w_{11} \mapsto \frac{1}{\delta_{12}}w_{11},\ w_{12} \mapsto \frac{1}{\delta_{12}}w_{12}$$

allows one to take $\gamma_{11} = 0$ and $\delta_{12} = 1$.

Finally, in the case (10.3) we conclude, with similar arguments, that we may assume either $\gamma_{d1} = 0$ and $\delta_{d1} = 1$, or $\gamma_{d1} = 1$ and $\delta_{d1} = 0$. But the switch $v_{d1} \leftrightarrow w_{d1}$ reduces the number of the isomorphism classes of nilpotent Lie algebras of $(1, 2, n)$-type coming from (10.3) to one.

Summing up, we have just one isomorphism class for each positive integer n given through (10.1), with $\gamma_{11} = 0$ and $\delta_{12} = 1$, if n is odd and through (10.3) with $\gamma_{d1} = 0$ and $\delta_{d1} = 1$, if n is even. Thus, we can state

Theorem 5. *Any nilpotent Lie* k*-algebra* $\mathfrak{h}$ *of* $(1, 2, n)$*-type with* $n < 2|\mathsf{k}|$ *and* 1*-dimensional centre decomposes, as a vector space, into*

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{v} \oplus \mathfrak{w}$$

with $\mathfrak{h}' \oplus \mathfrak{v}$ *and* $\mathfrak{w}$ *Abelian subalgebras of* $\mathfrak{h}$ *and* $\dim_{\mathsf{k}} \mathfrak{v} + 1 = \dim_{\mathsf{k}} \mathfrak{w} = \frac{n+1}{2}$ *if n is odd,* $\dim_{\mathsf{k}} \mathfrak{v} = \dim_{\mathsf{k}} \mathfrak{w} = \frac{n}{2}$ *if n is even. More precisely, we have:*

Let n be odd. Then, there exist bases $\{x,\ v_0 := y\}$ *of* $\mathfrak{h}'$ *with* $x \in \mathfrak{z}(\mathfrak{h})$, $\{v_1, \ldots, v_{\frac{n-1}{2}}\}$ *of* $\mathfrak{v}$ *and* $\{w_0, \ldots, w_{\frac{n-1}{2}}\}$ *of* $\mathfrak{w}$ *such that*

$$[v_i, w_j] = X_{ij}x + Y_{ij}y \qquad \left(i, j = 0, \ldots, \tfrac{n-1}{2}\right)$$

with $X := (X_{ij}) = I_{\frac{n+1}{2}}$ *and* $Y_{ij} = 0$*, excepting* $Y_{10} = 1$*, i.e.*

$$Y := (Y_{ij}) = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 0 & 0 & & \\ & & \ddots & \ddots & \\ & & & 0 & 0 \end{pmatrix} \quad \left(\left(\frac{n+1}{2} \right) - \text{by} - \left(\frac{n+1}{2} \right) \right).$$

Let n be even. Then, there exist bases $\{x,\ v_0 := y\}$ *of* $\mathfrak{h}'$ *with* $x \in \mathfrak{z}(\mathfrak{h})$, $\{v_1, \ldots, v_{\frac{n}{2}}\}$ *of* $\mathfrak{v}$ *and* $\{w_1, \ldots, w_{\frac{n}{2}}\}$ *of* $\mathfrak{w}$ *such that*

$$[v_i, w_j] = X_{ij}x + Y_{ij}y \ \left(i = 0, \ldots, \frac{n}{2}, j = 1, \ldots, \frac{n}{2} \right)$$

with $X_{ij} = Y_{ij} = 0$, *excepting* $X_{00} = X_{j+1,j} = Y_{10} = 1$, *i.e.*

$$X := (X_{ij}) = \begin{pmatrix} 1 & 0 & & \\ 1 & 0 & \ddots & \\ 0 & 1 & \ddots & 0 \\ & \ddots & \ddots & 0 \\ & \ddots & 0 & 1 \end{pmatrix} \quad \left(\left(\frac{n+1}{2} \right) - \text{by} - \frac{n}{2} \right)$$

and

$$Y := (Y_{ij}) = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 0 & \ddots & \\ & \ddots & \ddots & 0 \\ & \ddots & 0 & 0 \end{pmatrix} \quad \left(\left(\frac{n+1}{2} \right) - \text{by} - \frac{n}{2} \right).$$

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